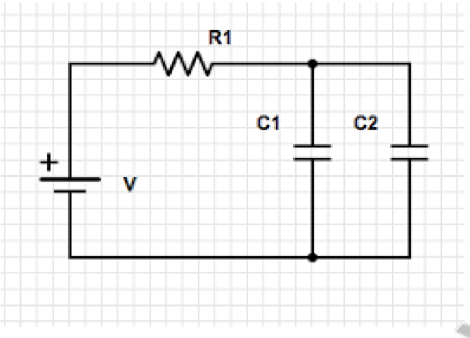
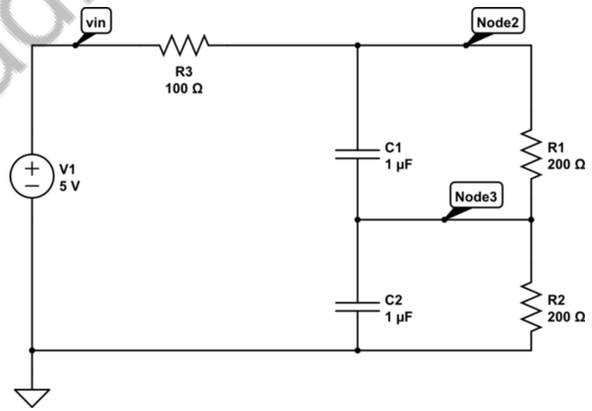
Electrical Science: 2021-22
Lecture 11
Second Order Circuits

By Dr. Sanjay Vidhyadharan

Second Order Circuit

- **First Order Circuit:** Any circuit with a single energy storage element, an arbitrary number of sources, and an arbitrary number of resistors is a circuit of order 1.





- **Second Order Circuit:**
  - 2nd -order circuit responses are described by 2nd -order differential equations

Second Order Circuit

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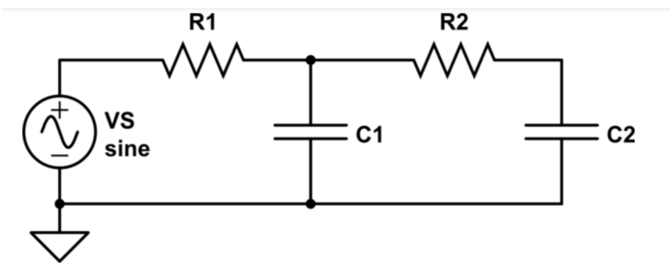
  - 2^{nd} -order circuit responses are described by 2^{nd} -order differential equations

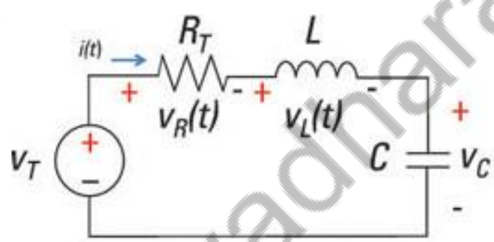
Order of a circuit	=	Order of the differential equation (DE) required to describe the circuit	=	The number of independent* energy storage elements (C's and L's)

* C's and L's are independent if they cannot be combined with other C's and L's (in series or parallel, for example)

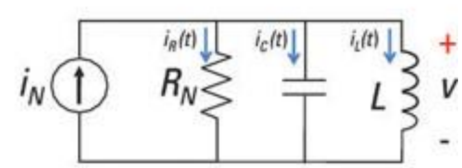
Second Order Circuits

- **Second Order Circuit:**
  - 2nd -order circuit responses are described by 2nd -order differential equations



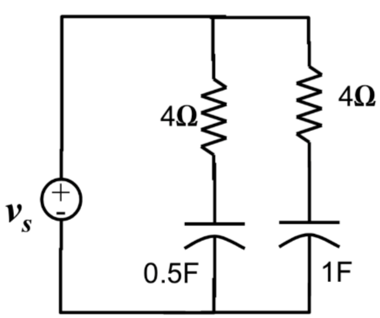


RLC series circuit



RLC parallel circuit

$$\frac{dV_{C1}}{dt} = \frac{dV_{C2}}{dt} + R_2C_2\frac{d^2V_{C2}}{dt^2}$$



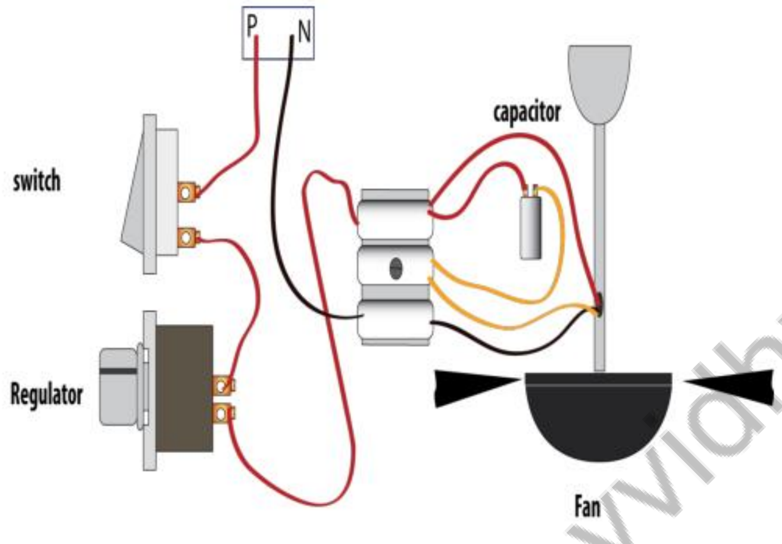
$$Z_1 = R_1 + \frac{1}{sC_1} = \frac{sC_1R_1 + 1}{sC_1}$$

$$Z_2 = R_2 + \frac{1}{sC_2} = \frac{sC_2R_2 + 1}{sC_2}$$

$$Z_1||Z_2 = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2}} = \frac{C_1C_2R_1R_2s^2 + (C_1R_1 + C_2R_2)s + 1}{C_1C_2(R_1 + R_2)s^2 + (C_1 + C_2)s}$$

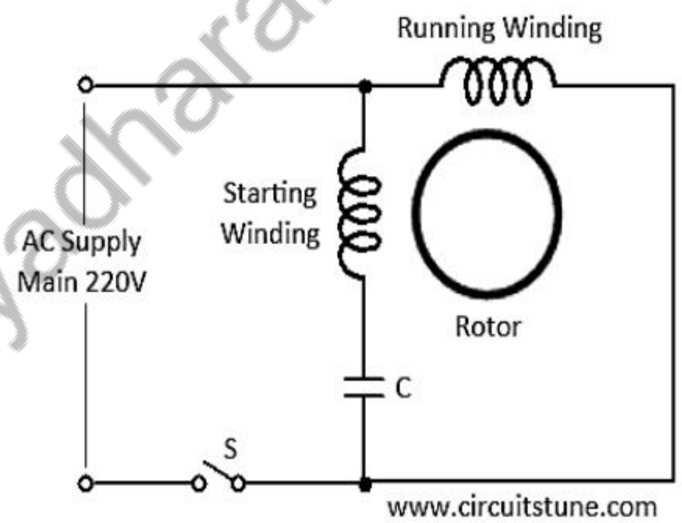
Applications of Second Order Circuits

Application of RC/RL/RLC



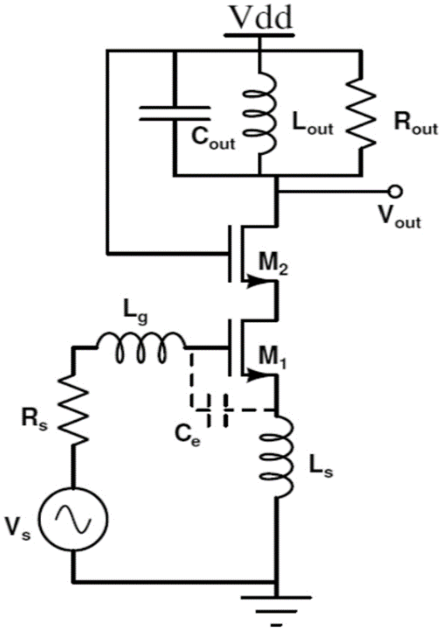
Electic Fan Circuit

Electric Fan

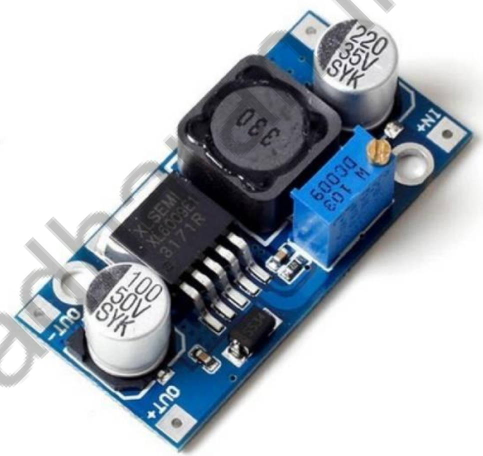


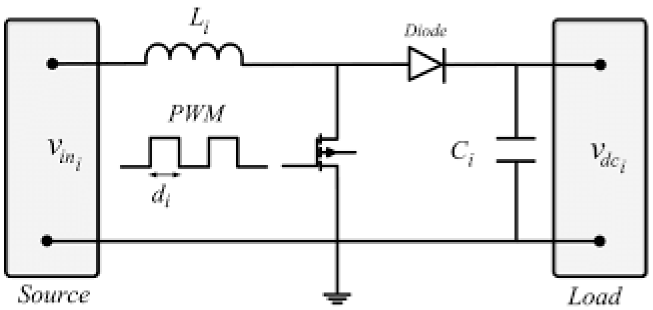
Ceiling fan wiring diagram

Applications of Second Order Circuits



Tunned Circuits





DC Boost Converters

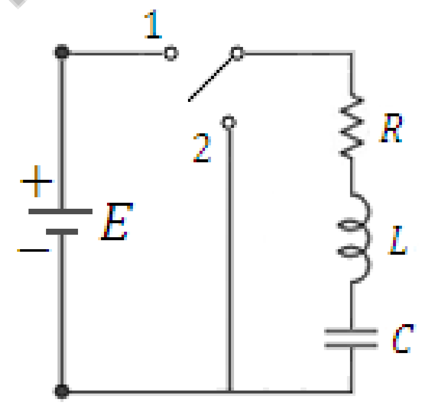
Response of a Series RLC Circuit

$$v_c + Ri + L\frac{di}{dt} = 0$$

Inserting $i = C\frac{dv}{dt}$;

$$v_c + RC\frac{dv}{dt} + LC\frac{d^2v}{dt^2} = 0$$

$$\gg \frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

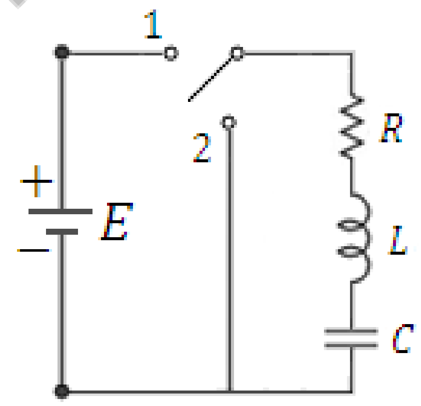


$$\gg \frac{d^2y(t)}{dt^2} + 2\alpha\frac{dy(t)}{dt} + \omega_n^2 y(t) = 0$$

$$\alpha = \frac{R}{2L};\ \omega_n = \frac{1}{\sqrt{LC}};$$

Response of a Series RLC Circuit

» $$\frac{d^2y(t)}{dt^2} + 2\alpha\frac{dy(t)}{dt} + \omega_n{}^2 y(t) = 0$$



$$s^2 + 2\alpha s + \omega_n^2 = 0$$

By Quadratic formula

$$s = \frac{-2\alpha \pm \sqrt{4\alpha^2 - 4\omega_n^2}}{2} = -\alpha \pm \sqrt{\alpha^2 - \omega_n^2}$$

$$\alpha = \frac{R}{2L};\ \omega_n = \frac{1}{\sqrt{LC}};$$

The two values of s

$$s_1 = -\alpha - \sqrt{\alpha^2 - \omega_n^2}$$

$$s_2 = -\alpha + \sqrt{\alpha^2 - \omega_n^2}$$

$$y(t) = A_1e^{s_1t} + A_2e^{s_2t}$$

(A_1 *and* A_2 are arbitrary constants and are determined from the initial conditions)

Response of a Series RLC Circuit

$y(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

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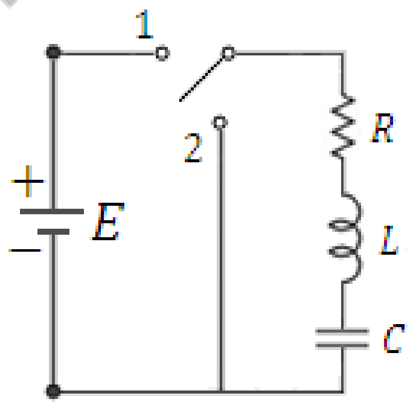


Three types of solutions are inferred:
1. If α > ω_n , we have the over-damped case.
2. If α = ω_n , we have the critically-damped case.
3. If α < ω_n , we have the under-damped case.

$$\alpha = \frac{R}{2L};\ \omega_n = \frac{1}{\sqrt{LC}};$$

- **Overdamped Case (α > ω_0)**
  α > ω_0 implies $R^2 > 4L/C$

$$\alpha = \frac{R}{2L};\ \omega_0 = \frac{1}{\sqrt{LC}}$$

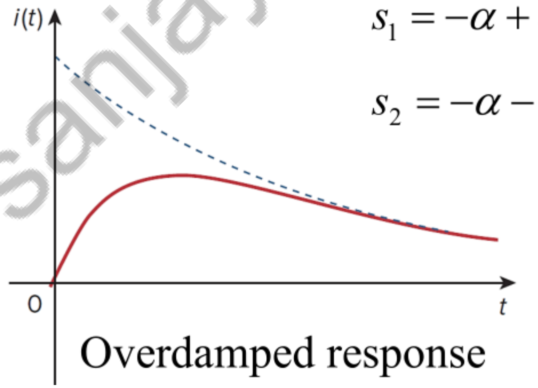
When this happens, both roots s_1 and s_2 are negative and real.

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

The response is given as,

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

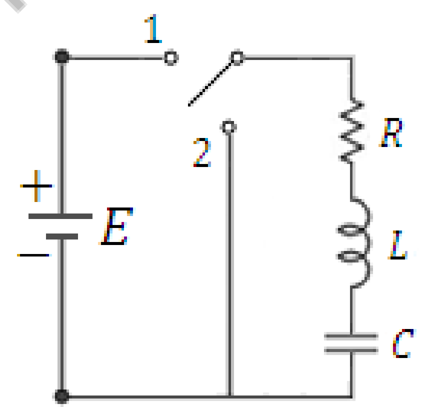


Overdamped response

Response of a Series RLC Circuit

$$y(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

(A_1 and A_2 arearbitraryconstants and are determined from the initial conditions)

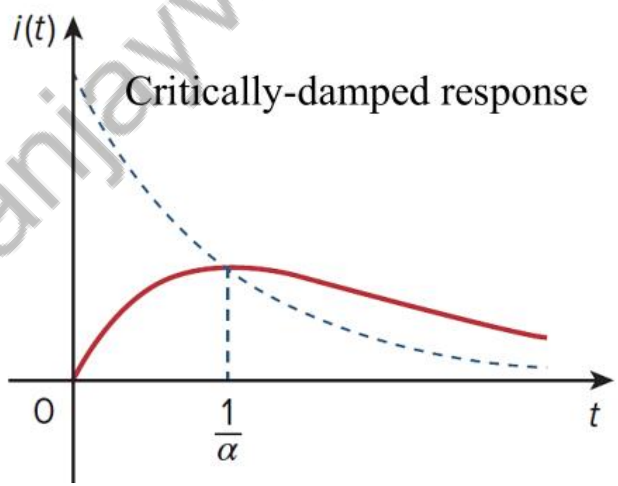


Three types of solutions are inferred:
1. If $\alpha > \omega_n$, we have the over-damped case.
2. If $\alpha = \omega_n$, we have the critically-damped case.
3. If $\alpha < \omega_n$, we have the under-damped case.

$$\alpha = \frac{R}{2L};\ \omega_n = \frac{1}{\sqrt{LC}};$$

- **Critically Damped Case ($\alpha = \omega_0$)**
  $\alpha = \omega_0$ implies $R^2 = 4L/C$ Thus $s_1 = s_2 = -\alpha = -\frac{R}{2L}$

$$i = (A_1 t + A_2)e^{-\alpha t}$$



Response of a Series RLC Circuit

Three types of solutions are inferred:
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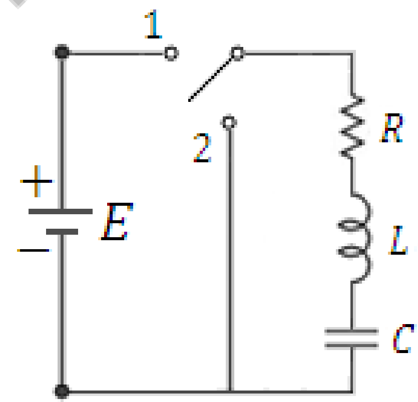


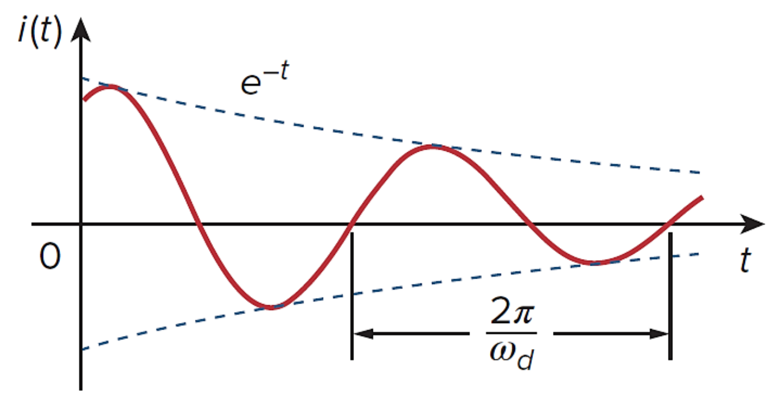
- **Underdamped Case ($\alpha < \omega_0$)**
  $\alpha < \omega_0$ implies $R^2 < 4L/C$. The roots may be written

$$s_1 = -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha + j\omega_d$$
$$s_2 = -\alpha - \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha - j\omega_d$$

where, $j = \sqrt{-1}$; $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

$$\alpha = \frac{R}{2L};\ \omega_n = \frac{1}{\sqrt{LC}};$$

$$i(t) = A_1 e^{-(\alpha - j\omega_d)t} + A_2 e^{-(\alpha + j\omega_d)t} = e^{-\alpha t}(A_1 e^{-j\omega_d t} + A_2 e^{-j\omega_d t})$$

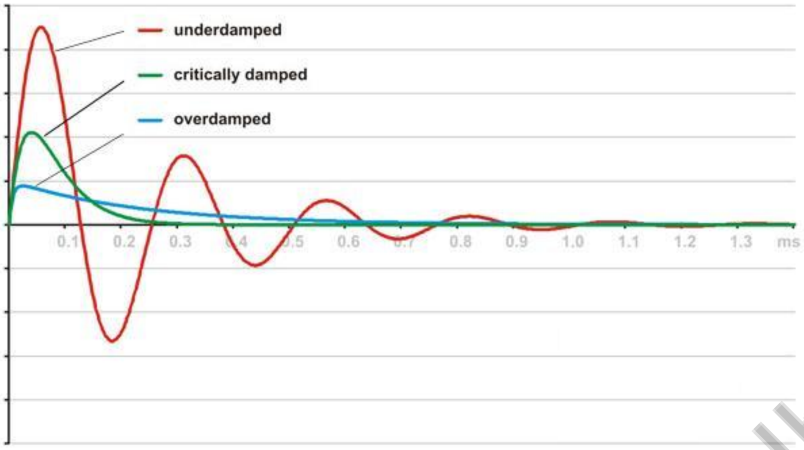


$$i(t) = e^{-\alpha t}[A_1(\cos\omega_d t + j\sin\omega_d t) + A_2(\cos\omega_d t - j\sin\omega_d t)]$$
$$= e^{-\alpha t}[(A_1 + A_2)\cos\omega_d t + j(A_1 - A_2)\sin\omega_d t]$$

$$i(t) = e^{-\alpha t}[B_1\cos\omega_d t + B_2\sin\omega_d t]$$

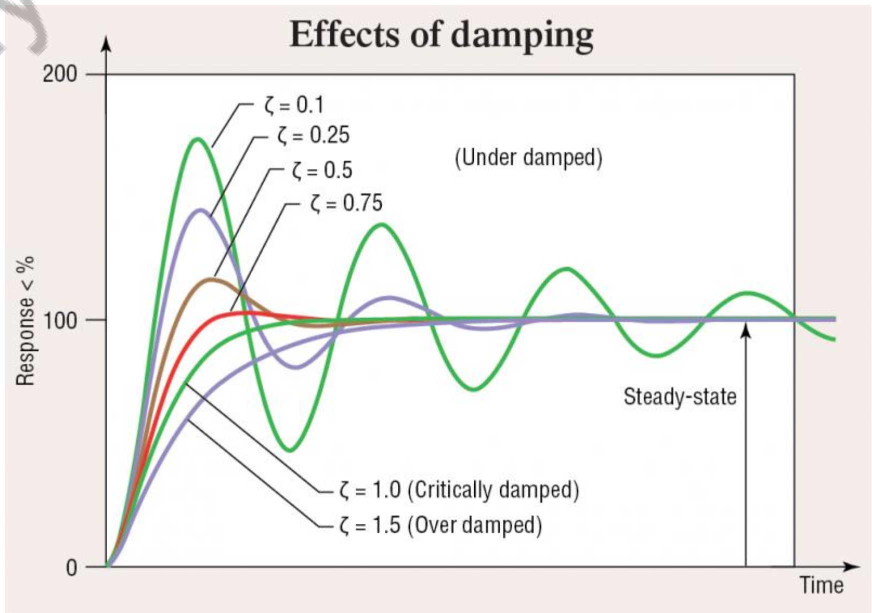
Response of a Series RLC Circuit

$$\alpha = \frac{R}{2L};\ \omega_n = \frac{1}{\sqrt{LC}};$$



$$\zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$$

$$\text{Damping Ratio} = \frac{Damping\ factor\ (\alpha)}{Natural\ Frequency\ (\omega_n)}$$



Response of a Series RLC Circuit

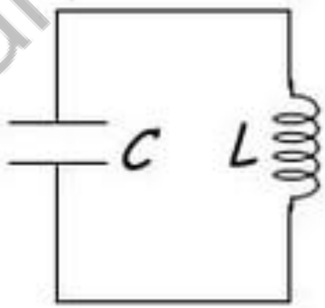
Special case if R=0

Kirchhoff: $V_C + V_L = 0$

$$\frac{Q}{C} + L\frac{dI}{dt} = 0$$

$$\frac{Q}{C} + L\frac{d^2Q}{dt^2} = 0$$

$$\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0$$



This is the SHM equation!!

$$\frac{d^2Q}{dt^2} + \omega^2 Q = 0$$ with $$\omega = \frac{1}{\sqrt{LC}}$$

Solution:

$$Q = Q_0 \cos(\omega t + \varphi)$$

Response of a Series RLC Circuit

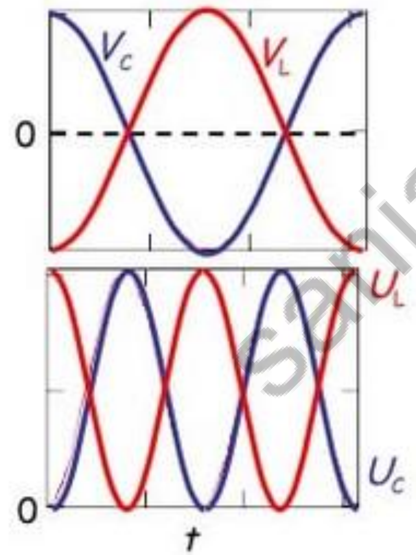
Special case if R=0

$$Q = Q_0 \cos(\omega t + \varphi) \qquad I = -Q_0 \omega \sin(\omega t + \varphi)$$

$$V_C = \frac{Q}{C} = \frac{Q_0}{C}\cos(\omega t + \varphi) \qquad V_L = L\frac{dI}{dt} = -LQ_0\omega^2 \cos(\omega t + \varphi)$$

$$U_C = \frac{Q^2}{2C} = \frac{Q_0^2}{2C}\cos^2(\omega t + \varphi) \qquad U_L = \frac{1}{2}LI^2 = \frac{1}{2}LQ_0^2\omega^2 \sin^2(\omega t + \varphi) = \frac{Q_0^2}{2C}\sin^2(\omega t + \varphi)$$

$$\frac{LQ_0^2\omega^2}{2} = \frac{LQ_0^2}{2}\frac{1}{LC} = \frac{Q_0^2}{2C}$$

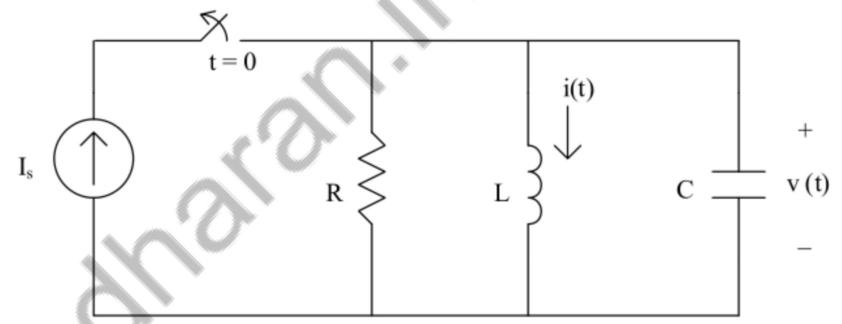


Energy is continuously transformed: electric ↔ magnetic

Compare to mass and spring (SHM): kinetic ↔ potential

Response of a Parallel RLC Circuit

Assume initial inductor current I_0 and initial capacitor voltage V_0,



Three elements are in parallel, they have the same voltage v across them. Applying KCL at the top node gives,

$$\frac{v}{R}+\frac{1}{L}\int_{-\infty}^{0} v(\tau)d\tau + C\frac{dv}{dt}=0$$

$$\frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{1}{LC}v=0$$

$$s^2+\frac{1}{RC}s+\frac{1}{LC}=0$$

$$s_{1,2}=\frac{1}{2RC}\pm\sqrt{\left(\frac{1}{2RC}\right)^2-\frac{1}{LC}}$$

$$s_{1,2}=-\alpha\pm\sqrt{\alpha^2-\omega_0^2}\ , \quad \alpha=\frac{1}{2RC}, \quad \omega_0=\frac{1}{\sqrt{LC}}$$

Response of a Parallel RLC Circuit

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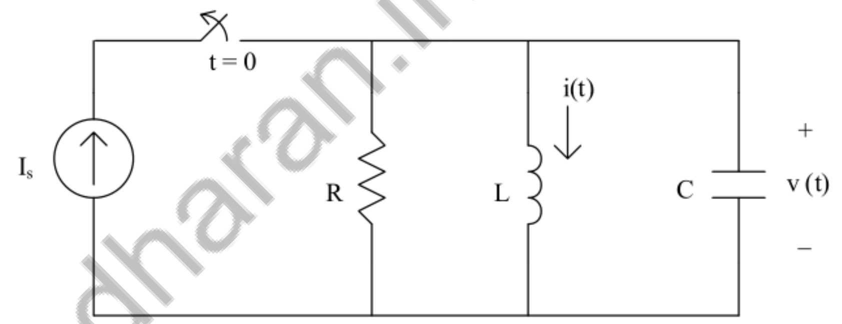

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$$\frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{1}{LC}v=0$$

$$s^2+\frac{1}{RC}s+\frac{1}{LC}=0$$

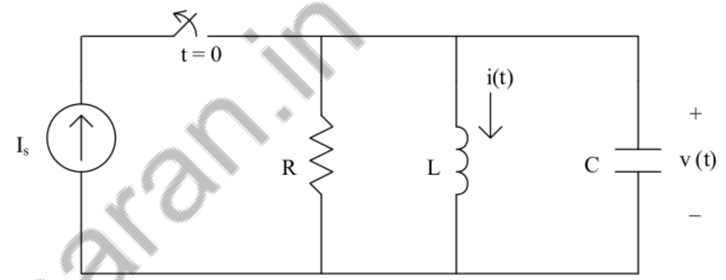
$$s_{1,2}=\frac{1}{2RC}\pm\sqrt{\left(\frac{1}{2RC}\right)^2-\frac{1}{LC}}$$

$$s_{1,2}=-\alpha\pm\sqrt{\alpha^2-\omega_0^2}\ ,\quad \alpha=\frac{1}{2RC},\quad \omega_0=\frac{1}{\sqrt{LC}}$$

Response of a Parallel RLC Circuit

$$s_{1,2} = \frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

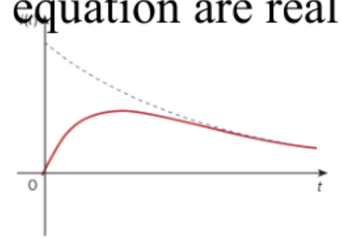
$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$,	$\alpha = \frac{1}{2RC}$,	$\omega_0 = \frac{1}{\sqrt{LC}}$



Overdamped Case ($\alpha > \omega_0$)

$\alpha > \omega_0$ => L/C > 4R^2. The roots of the characteristic equation are real and negative. The response is,

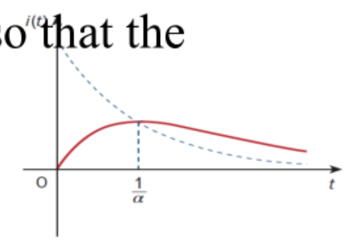
$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$



Critically Damped Case ($\alpha = \omega_0$)

$\alpha = \omega_0$ => L/C = 4R^2. The roots are real and equal so that the response is,

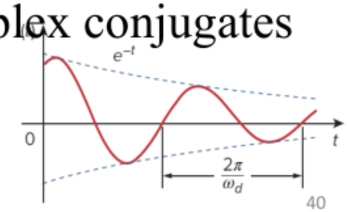
$$v(t) = (A_1 + A_2\, t)e^{-\alpha t}$$



Underdamped Case ($\alpha < \omega_0$)

$\alpha < \omega_0$ => L/C < 4R^2. In this case the roots are complex conjugates expressed as $s_{1,2} = -\alpha \pm j\omega_d$; $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

$$v(t) = e^{-\alpha t}(A_1 \cos\omega_d t + A_2 \sin\omega_d t)$$



Thank you