Electrical Science: 2021-22

Lecture 16

Resonance in AC Circuits

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Resonance in RLC Series AC Circuits

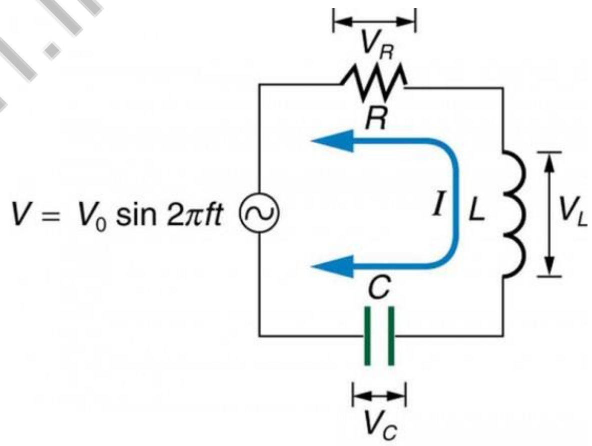
$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

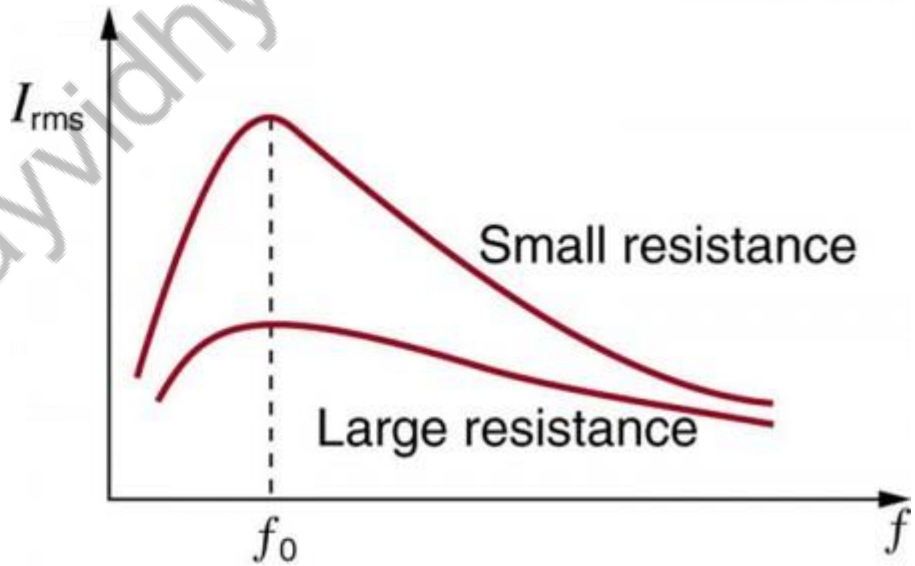
$Z_{min} = R$ at Resonant Frequency where $X_L = X_C$

$I_{max} = V/R$

$$2\pi f_0 L = \frac{1}{2\pi f_0 C}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$





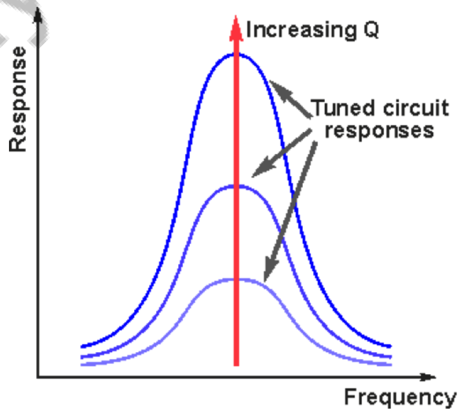
QUALITY FACTOR (Q)

The "sharpness" of the resonance in a resonant circuit is measured quantitatively by the quality factor Q.

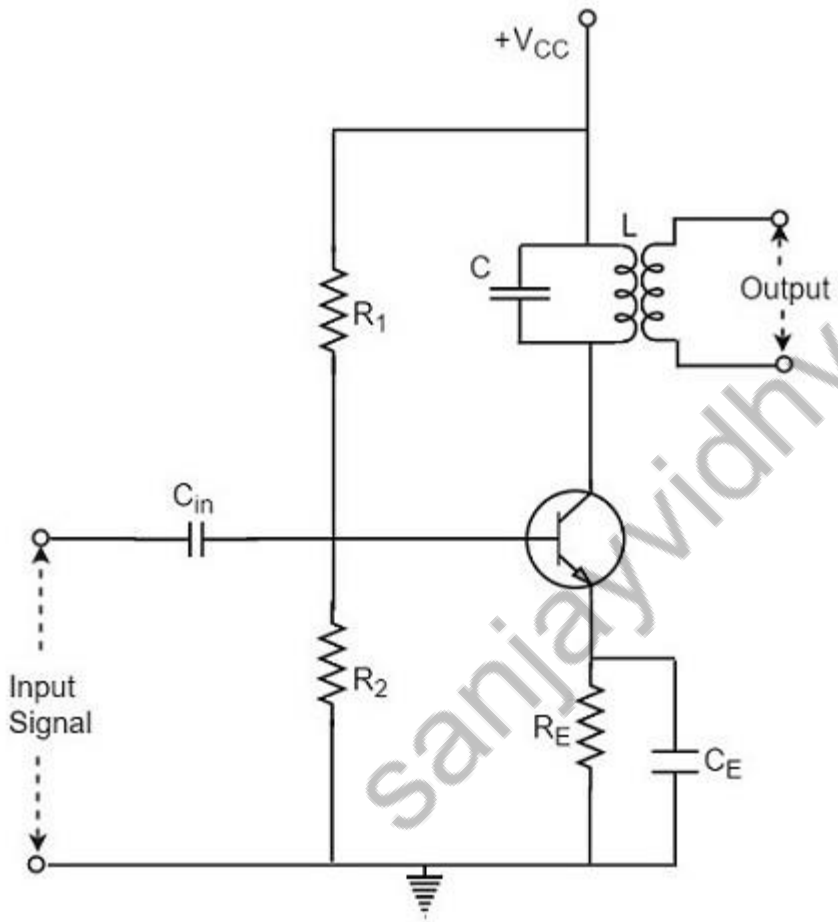
• The quality factor relates the maximum or peak energy stored to the energy dissipated in the circuit per cycle of oscillation:

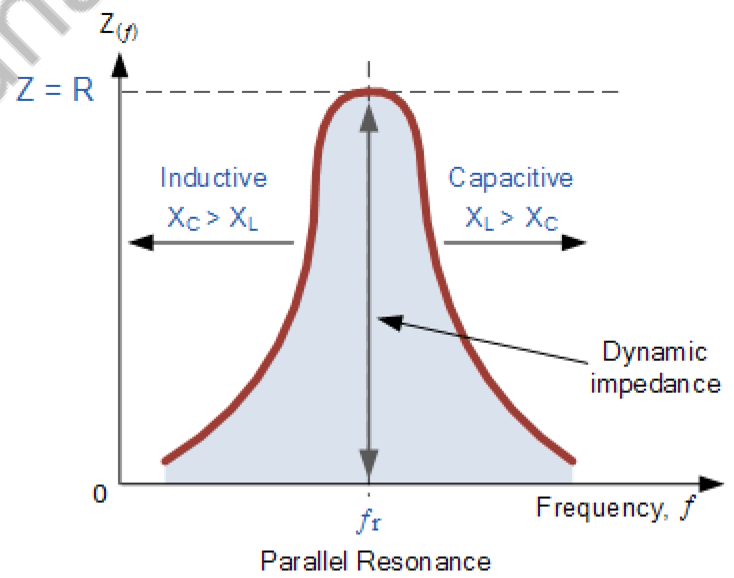
$$Q = 2\pi\left(\frac{\text{Peak energy stored in the circuit}}{\text{Energy dissipated by the circuit in one period at resonance}}\right)$$

• It is also regarded as a measure of the energy storage property of a circuit in relation to its energy dissipation property.



Tunned Circuits



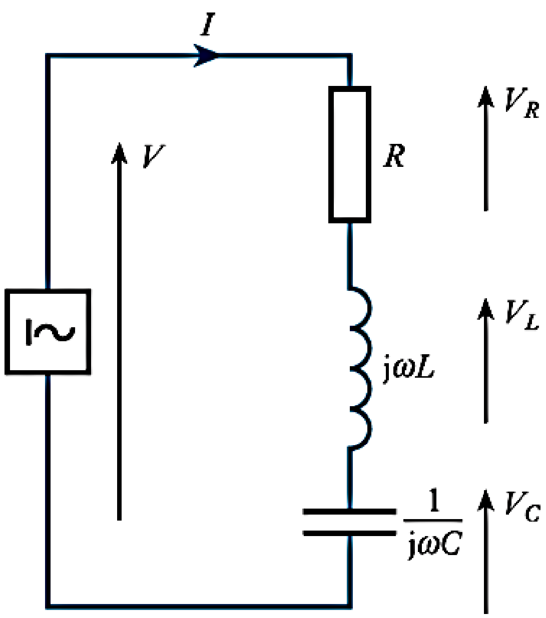


QUALITY FACTOR (Q)

In the series *RLC* circuit, the quality factor (*Q*) is,

$$Q = 2\pi \left(\frac{\frac{1}{2} L I^2}{\frac{1}{2} I^2 R \left(\frac{1}{f_r} \right)} \right) = \frac{2\pi f_r L}{R}$$

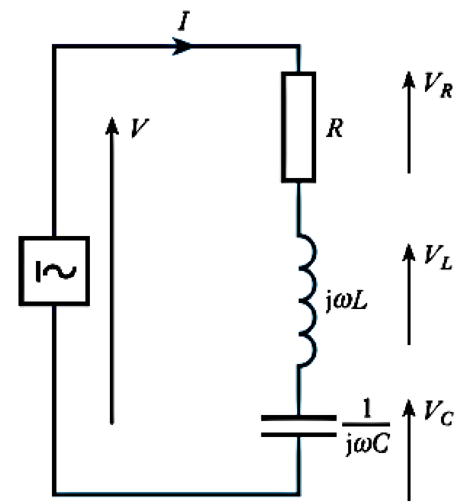
$$Q = \frac{\omega_r L}{R} = \frac{1}{\omega_r C R} = \frac{1}{R}\sqrt{\frac{L}{C}}$$



QUALITY FACTOR (Q)

- The Q factor is also defined as the ratio of the reactive power, of either the capacitor or the inductor to the average power of the resistor at resonance:

$$Q = \left(\frac{\text{Reactive power}}{\text{Average power}} \right)$$



- For inductive reactance X_L at resonance:

$$Q = \left(\frac{\text{Reactive power}}{\text{Average power}} \right) = \frac{I^2 X_L}{I^2 R} = \frac{\omega_r L}{R}$$

- For capacitive reactance X_L at resonance:

$$Q = \left(\frac{\text{Reactive power}}{\text{Average power}} \right) = \frac{I^2 X_C}{I^2 R} = \frac{1}{\omega_r C R}$$

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Resonance in RLC Series AC Circuits

The voltage across resistor at f_r is,

$$V_R = I_R \times R = I_m \times R = \frac{V}{R} \times R \Rightarrow V_R = V$$

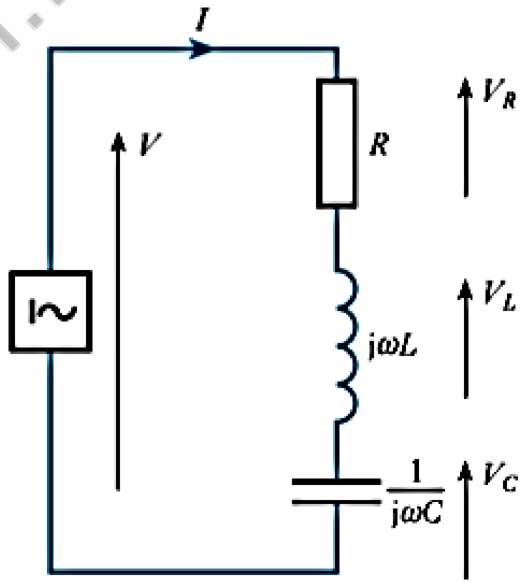
The voltage across inductor at f_r is,

$$|V_L| = X_L \times I_L = \omega_r L \times I_m = \omega_r L \times \frac{V}{R} = \frac{\omega_r L}{R} V = QV$$

$$\Rightarrow |V_L| = QV$$

The voltage across capacitor at f_r is,

$$|V_C| = X_C \times I_C = \frac{1}{\omega_r C} \times I_m = \frac{1}{\omega_r C} \times \frac{V}{R} = \frac{1}{\omega_r CR} V = QV \Rightarrow |V_C| = QV$$

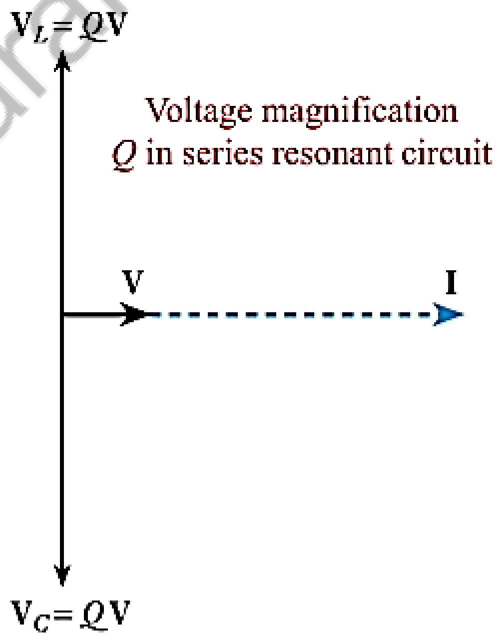


Resonance in RLC Series AC Circuits

- Q is termed as Q factor or voltage magnification, because V_C or V_L equals Q multiplied by the source voltage V.

- In a series RLC circuit, values of VL and VC can actually be very large at resonance and can lead to component damage if not recognized and subject to careful design.

$$Q = \frac{\omega_r L}{R} = \frac{1}{\omega_r C R} = \frac{1}{R}\sqrt{\left(\frac{L}{C}\right)}$$



BANDWIDTH AND HALF POWER FREQUENCIES

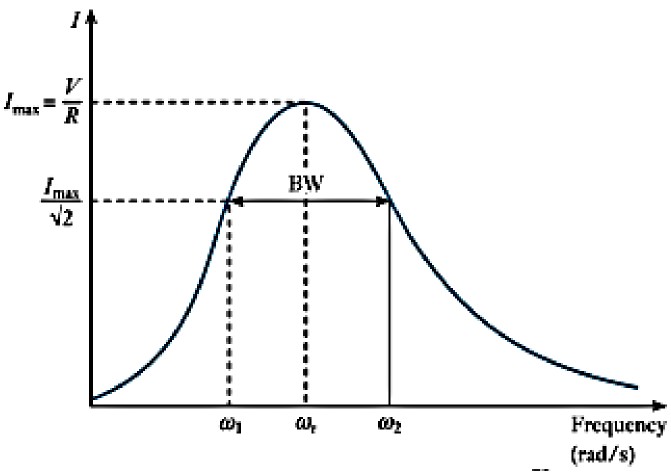
In a series *RLC* circuit, at resonance, maximum power is drawn. i.e.,

$$P_r = I_{\max}{}^2 \times R; \text{ where } I_{\max} = \frac{V}{R} \text{ at resonance}$$

Bandwidth represents the range of frequencies for which the power level in the signal is at least half of the maximum power.

$$\frac{P_r}{2} = \frac{I_{\max}{}^2 \times R}{2} = \left(\frac{I_{\max}}{\sqrt{2}}\right)^2 \times R$$

The bandwidth of a circuit is also defined as the frequency range between the half-power points when $I = I_{max}/\sqrt{2}$.



BANDWIDTH AND HALF POWER FREQUENCIES

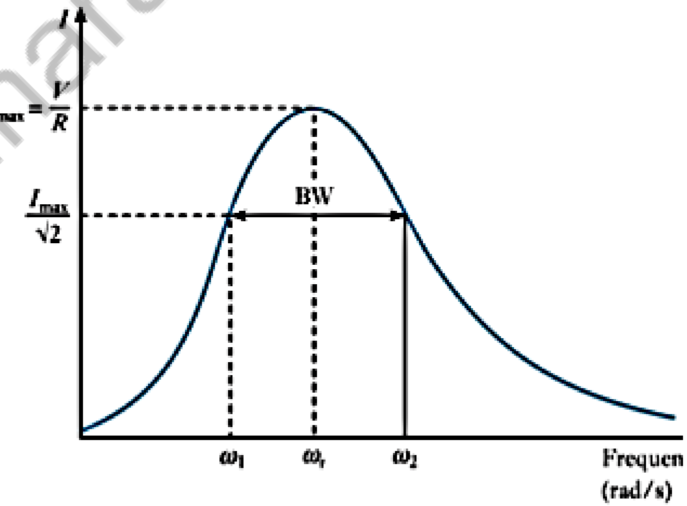
Thus, the condition for half-power is given when

$$\left|I\right| = \frac{I_{\max}}{\sqrt{2}} = \frac{V}{R\sqrt{2}}$$

The vertical lines either side of $|I|$ indicate that only the magnitude of the current is under consideration – but the phase angle will not be neglected.

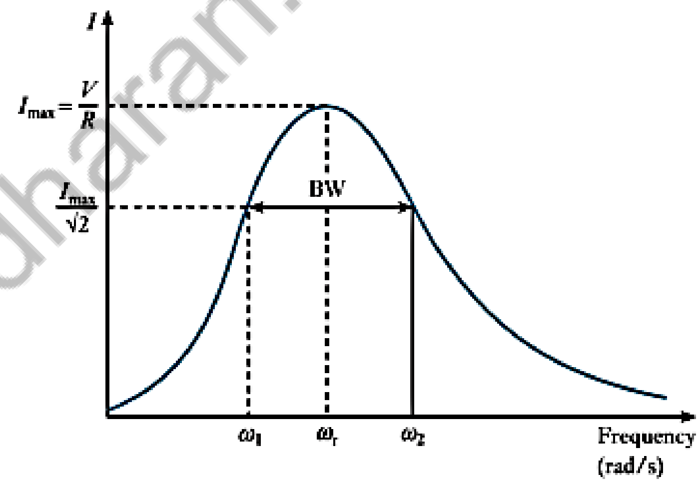
The impedance corresponding to half power-points including phase angle is

$$Z(\omega_{1,2}) = R\sqrt{2}\angle \pm 45^{\circ}$$



The resonance peak, bandwidth and half-power frequencies

BANDWIDTH AND HALF POWER FREQUENCIES



- The impedance in the complex form

$$Z(\omega_{1,2}) = R(1 \pm j1)$$

- Thus for half power,

$$I = \frac{V}{R(1 \pm j1)} \quad \text{and} \quad Z = R(1 \pm j1)$$

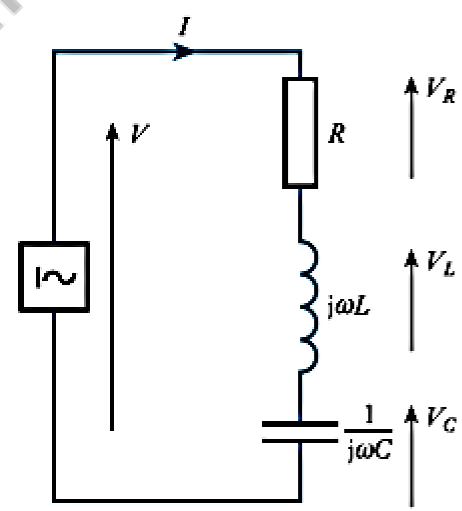
- At the half-power points, the phase angle of the current is 45°. Below the resonant frequency, at ω_1, the circuit is capacitive and $Z(\omega_1) = R(1 - j1)$.
- Above the resonant frequency, at ω_2, the circuit is inductive and

$Z(\omega_2) = R(1 + j1)$.

BANDWIDTH AND HALF POWER FREQUENCIES

Now, the circuit impedance is given by,

$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right) = R\left(1 + j\left(\frac{\omega L}{R} - \frac{1}{\omega CR}\right)\right)$$



At half power points, $Z = R(1 \pm j1)$

By comparison of above two equations, resulting in

$$\frac{\omega L}{R} - \frac{1}{\omega CR} = \pm 1$$

As we know,

$$Q = \frac{\omega_r L}{R} = \frac{1}{\omega_r CR}$$

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BANDWIDTH AND HALF POWER FREQUENCIES

- Now, by multiplying and dividing with ω_r:

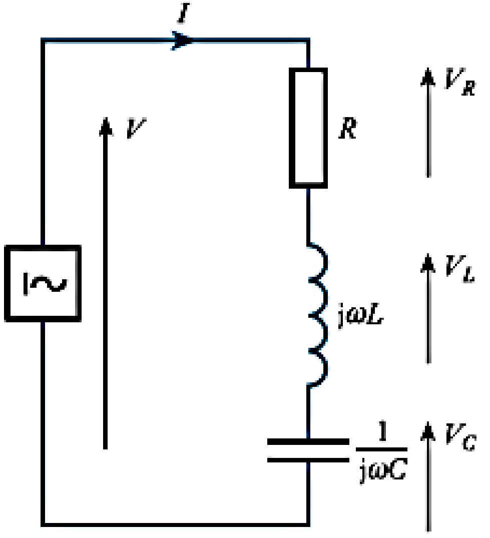
$$\frac{\omega L}{R}\frac{\omega_r}{\omega_r}-\frac{1}{\omega CR}\frac{\omega_r}{\omega_r}=\pm 1 \Rightarrow \frac{\omega}{\omega_r}Q-\frac{\omega_r}{\omega}Q=\pm 1 \Rightarrow Q\left(\frac{\omega}{\omega_r}-\frac{\omega_r}{\omega}\right)=\pm 1$$

- For ω_2:

$$Q\left(\frac{\omega_2}{\omega_r}-\frac{\omega_r}{\omega_2}\right)=1$$

- For ω_1:

$$Q\left(\frac{\omega_1}{\omega_r}-\frac{\omega_r}{\omega_1}\right)=-1$$

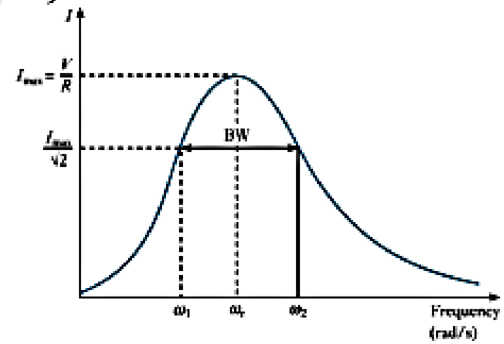


BANDWIDTH AND HALF POWER FREQUENCIES

The half-power frequencies ω_2 and ω_1 are obtained as,

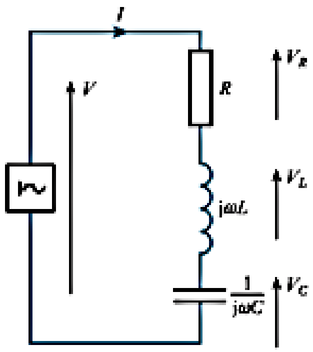
$$\omega_2 = \frac{\omega_r}{2Q} + \omega_r\sqrt{1+\frac{1}{4Q^2}}$$

$$\omega_1 = \frac{-\omega_r}{2Q} + \omega_r\sqrt{1+\frac{1}{4Q^2}}$$



The bandwidth is obtained as:

$$\text{BW} = \omega_2 - \omega_1 = \frac{\omega_r}{Q}$$ i.e. $$\text{Bandwidth} = \frac{\text{Resonant frequency}}{Q \text{ factor}}$$



Resonant frequency in terms of ω_2 and ω_1, is expressed as:

$$\omega_r = \sqrt{\omega_1\omega_2}$$

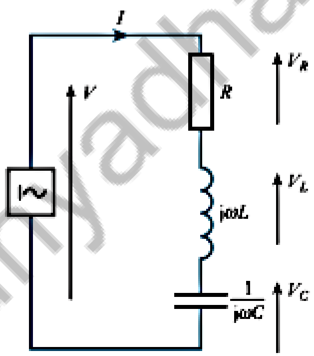
BANDWIDTH AND HALF POWER FREQUENCIES

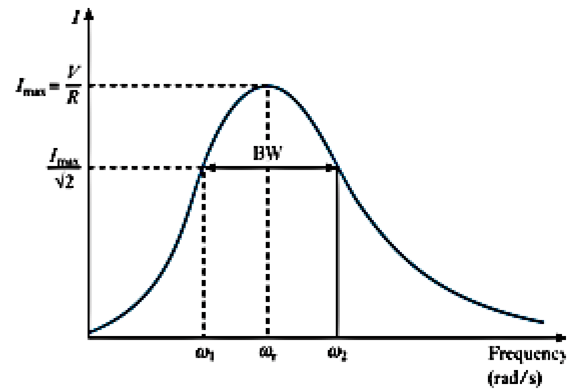
The bandwidth is also expressed as:

$$\omega_2 - \omega_1 = \frac{\omega_r}{Q} \Rightarrow \omega_2 - \omega_1 = \frac{R}{L} \text{ rad/s}$$

(or)

$$f_2 - f_1 = \frac{R}{2\pi L} \text{ Hz}$$





For $Q >> 1$,

$$\omega_r - \omega_1 \approx \frac{\text{BW}}{2} \Rightarrow \omega_1 = \omega_r - \frac{\text{BW}}{2} \Rightarrow \omega_1 = \omega_r - \frac{\text{R}}{2L} \text{ rad/s}$$

$$\omega_2 - \omega_r \approx \frac{\text{BW}}{2} \Rightarrow \omega_2 = \omega_r + \frac{\text{BW}}{2} \Rightarrow \omega_2 = \omega_r + \frac{\text{R}}{2L} \text{ rad/s}$$

Calculating the Power

Example 1 For a *RLC* series circuit having a 40.0 Ω resistor, a 3.00 mH inductor, a 5.00 μF capacitor, and a voltage source with a V_{rms} of 120 V. Find the average power at the circuit's resonant frequency.

, $P_{ave} = (3.00\text{ A})(120\text{ V})(1) = 360\text{ W}$ at resonance (1.30 kHz)

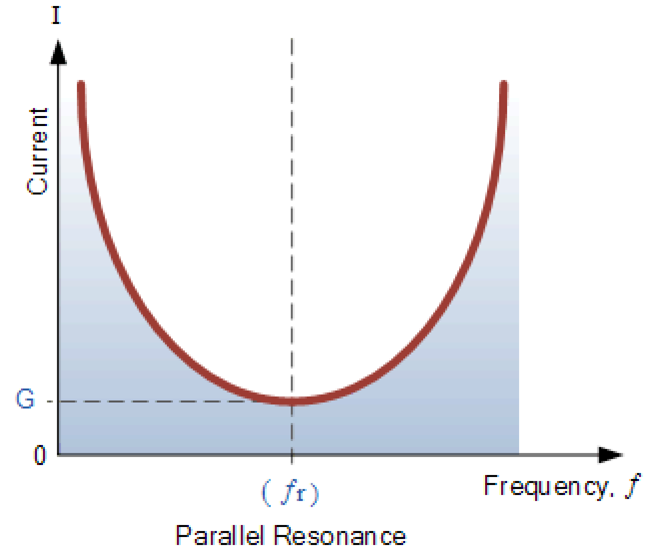
Resonance in Parallel RLC Circuit

As the total susceptance is zero at the resonant frequency, the admittance is at its minimum and is equal to the conductance, G. Therefore at resonance the current flowing through the circuit must also be at its minimum as the inductive and capacitive branch currents are equal ($I_L = I_C$) and are 180° out of phase.

$$I_R = \frac{V}{R}$$

$$I_L = \frac{V}{X_L} = \frac{V}{2\pi f L}$$

$$I_C = \frac{V}{X_C} = V.2\pi f C$$



Therefore, $I_T = \text{vector sum of } (I_R + I_L + I_C)$

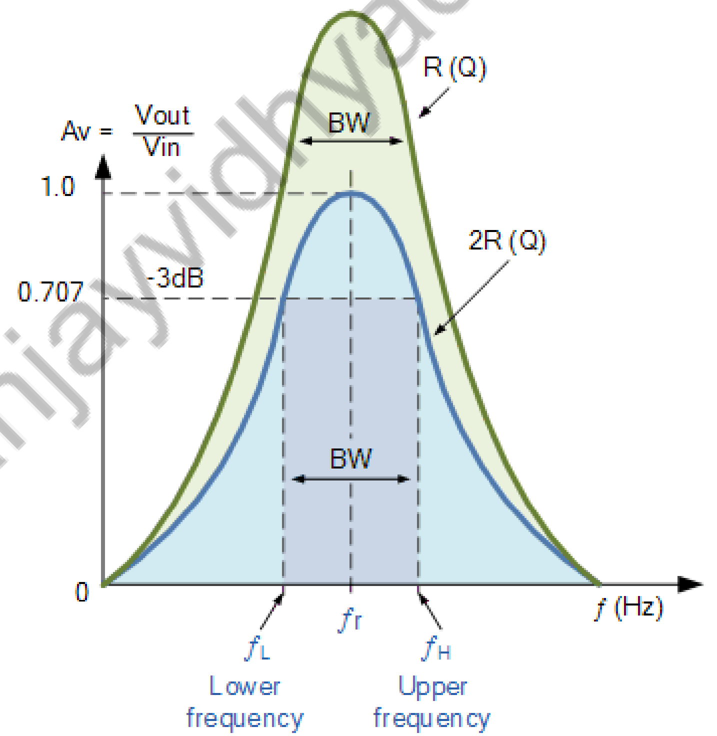
$$I_T = \sqrt{I_R^2 + (I_L + I_C)^2}$$

$$I_T = \sqrt{I_R^2 + 0^2} = I_R$$

Resonance in Parallel RLC Circuit

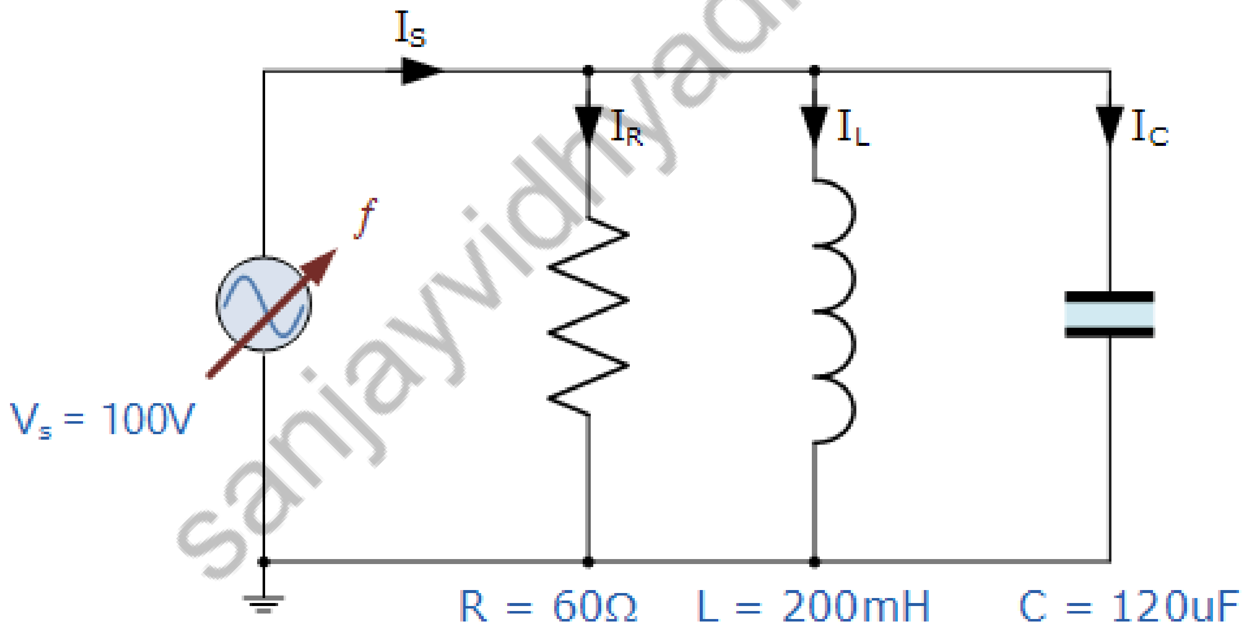
$$BW = f_r / Q \quad \text{or} \quad BW = f_{upper} - f_{lower}$$

$$\text{Quality Factor, } Q = \frac{R}{2\pi f L} = 2\pi f CR = R\sqrt{\frac{C}{L}}$$



Resonance in Parallel RLC Circuit

Example 2: A parallel resonance network consisting of a resistor of 60Ω, a capacitor of 120uF and an inductor of 200mH is connected across a sinusoidal supply voltage which has a constant output of 100 volts at all frequencies. Calculate, the resonant frequency, the quality factor and the bandwidth of the circuit, the circuit current at resonance and current magnification.



Resonance in Parallel RLC Circuit

Example 2: 1. Resonant Frequency, f_r

$$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.2.120.10^{-6}}} = 32.5\text{Hz}$$

2. Inductive Reactance at Resonance, X_L

$$X_L = 2\pi f L = 2\pi.32.5.0.2 = 40.8\Omega$$

3. Quality factor, Q

$$Q = \frac{R}{X_L} = \frac{R}{2\pi f L} = \frac{60}{40.8} = 1.47$$

4. Bandwidth, BW

$$BW = \frac{f_r}{Q} = \frac{32.5}{1.47} = 22\text{Hz}$$

5. The upper and lower -3dB frequency points, f_H and f_L

$$f_L = f_r - \frac{1}{2}BW = 32.5 - \frac{1}{2}(22) = 21.5\text{Hz}$$

$$f_H = f_r + \frac{1}{2}BW = 32.5 + \frac{1}{2}(22) = 43.5\text{Hz}$$

Resonance in Parallel RLC Circuit

Example 2:

6. Circuit Current at Resonance, I_T

At resonance the dynamic impedance of the circuit is equal to R

$$I_T = I_R = \frac{V}{R} = \frac{100}{60} = 1.67A$$

7. Current Magnification, I_{mag}

$$I_{MAG} = Q \times I_T = 1.47 \times 1.67 = 2.45A$$

Note that the current drawn from the supply at resonance (the resistive current) is only 1.67 amps, while the current flowing around the LC tank circuit is larger at 2.45 amps. We can check this value by calculating the current flowing through the inductor (or capacitor) at resonance.

$$I_L = \frac{V}{X_L} = \frac{V}{2\pi f L} = \frac{100}{2\pi.32.5.0.2} = 2.45A$$

Thank you